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Dedicated to mathematics in general and to the following aims in particular: (1) a study of the common problems of secondary and collegiate mathematics teaching, (2) a true valuation of the disciplines of mathematics, (3) the publication of high class expository papers on mathematics, (4) the development of greater public interest in mathematics by the publication of authoritative papers treating its cultural, humanistic and historical phases.

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SHADOW-LAND IN MATHEMATICS

Most epoch-creating mathematicians have lived long enough to sense the world's applause of their genius. Thus it was with Newton, with Descartes, with Laplace, and thus it is with Einstein today. But one has only to read history to know that the human side of not a few of the world's great mathematicians has been dark with the tragedy of bitter disappointment and unappreciation—a darkness that knew no breaking up to the hour of death.

Herman Grassmann's (1809-1877) calculus—almost as marvelous a creation as Hamilton's Quaternion Theory—made but little appeal at first even to minds like Gauss' and Mobius'.

**"Tired of waiting for appreciation, at the age of fifty-three, this wonderful man with a heavy heart gave up mathematics. . . . . Only in recent years has the wonderful richness of his discoveries begun to be appreciated."*

Niels Abel (1802-1829), Norwegian, now glorified as the creator of Abelian integrals, had a sensitiveness that must have fairly bled at the haughty indifference of Gauss to whom he had referred his proof that the general equation of the fifth degree is unsolvable.

Even more tragic to him must have been the silence of the French Academy following his submission to it of a memoir pronounced later by Jacobi to be "the greatest discovery of the century on the integral calculus." Long years before this recognition Abel was dead. For the living Abel the Academy's silence was never broken.

The Hungarian, Johann Bolyai, co-founder with the Russian Lobatchewksy, of non-Euclidean geometry, wrote in twenty-six pages his immortal "Science of Absolute Space", an achievement which remained for thirty-five years practically unnoticed by mathematicians. So, his glory, too, only came after he had passed on.—S. T. S.

*Cajori's History of Mathematics.

*THE PUBLIC USEFULNESS OF MATHEMATICS

By E. R. HEDRICK
University of California at Los Angeles

It is usual to emphasize the value of mathematics as a training for the whole mind, and to dwell upon the beauties and satisfactions that can be derived from the perfection of its logical proofs and from the grandeur of its higher concepts. It is also commonly known and commonly emphasized that a training in mathematics is essential for serious pursuit of engineering, of any of the physical sciences, or for some other highly technical pursuits. The history of the development of mathematics is in itself an interest to many, and it shows clearly the way in which the race has gradually conquered the physical and the quantitative problems which have arisen throughout history.

For the greater part of my time today, I am to speak to you regarding the broader phases of the usefulness of mathematical training to the whole public. Other speakers in this series will present to you in detail interesting phases of the topics that I have passed over hurriedly: one speaker will outline for you some of the interesting characters of ancient mathematics; another will speak of its applications in the sciences; still another will tell of the slow progress of our race in the development of the ideas of number and of quantity that now seem to us to be almost instinctive.

That number ideas and quantitative thinking are not at all instinctive, and that such things must be *taught* to children will be obvious to you if you will think, for example, of the clumsy Roman system of numerals, a notation of such a high civilization as that of ancient Rome. You might try, for instance, to do some simple arithmetic, say, to multiply 36 by 24, using the Roman numerals.

The management of numbers and of quantities is the chief purpose of mathematics. Training in thinking about quantities is held by the recent National Committee on the Reorganization of Mathematical Teaching to be the prime object of all teaching of mathematics in the schools. Those who dislike mathematics are sometimes bold enough to propose omitting it from school curricula. To them and to you, I wish to emphasize today not the history of mathematics, nor its beauties; not the need of mathematics for specialists or for engineers;

*Prepared for presentation over the radio, February 4, 1933.

but rather the public usefulness of mathematics, the need for a knowledge of mathematics by all the public.

Modern civilization has become more and more involved with quantities and with relations between quantities. Computation has replaced conjecture, in business as well as in engineering; in insurance as well as in physics; in economics as well as in astronomy; in the arts of war (artillery, airplanes, maps) and in the arts of peace (automobiles, bond-issues, business forecasts). Quantities now enter in the lives of all active people, in the businesses that they conduct, and in the public questions which require legislation, on which all the voters must record public opinion. Some of these public questions I shall detail.

A prime instance of the widespread occurrence of quantities in private and in public affairs, and of the management of those quantities by mathematical means, is the case of *money* and of *compound interest* on money. Who is there who is not affected both indirectly and directly? How many questions of public economics hinge upon a true understanding of the mathematics of this matter of compound interest? Thus thousands upon thousands are suffering today through some ill-conceived scheme of installment buying based on interest payments that are easy to calculate if they had only known. Misrepresentation of such plans is common. Interest rates that are actually as high as twenty per cent, forty per cent, fifty per cent, are charged under some plausible plan. The deceptions that have been practiced are based on public ignorance of simple relationships that are nothing else than the well-known geometric progressions taught in every high school. If the algebra has been well taught, the students known this, and can compute the real interest rates charged. For compound interest does work precisely in geometric progression, the original principal being multiplied each year (or each period) by 1 plus the rate of interest.

To leave all public questions that involve compound interest to the so-called "experts" is not a democratic means of solution of our problems. Such public questions are more extensive than the laws on usury, to which I have referred, and the associated questions of installment buying, small loans, second mortgages, and other plans for refinancing. Another large public question, for example, is that of life insurance, and the safeguarding of the public in that and in other forms of insurance. These questions, too, are based directly on com-

pound interest computations, and they should be more generally understood.

Bond issues, international debts, evaluaton of public-utility properties, unemployment insurance, pensions, are other vital public issues of today whose decision rests upon public opinion, but whose intelligent discussion is impossible without mathematical knowledge. The danger in every such quantitative public question is that emotion will over-ride reason, when the basis for reason—that is, the ability to think about quantities—is absent. The danger is that vague harangues will take the place of those accurate computations that are entirely possible. The danger is that ill-advised plans based on such emotional appeals in place of computations will invite renewed disaster. Pension schemes that hold rosy promises for old age will *fail* if they have no sound basis of calculation; calculation that is mathematically easy and not hard to understand by anyone who has mastered high-school algebra in a well-taught school.

Unemployment insurance is doomed in advance if it is not mathematically sound. Every quantitative public economic measure either is based on mathematical calculation, or else it is doomed to disaster. Public opinion guided only by oratorical vagueness may choose the wrong path. Any discrimination, any intelligent discussion, either pro or con, must of necessity state the elements of the mathematical foundation, or else it must risk false decision, wrong choice, fresh disaster.

Is the public "not interested" in the mathematical basis of which decisions must rest if they are to be sound? Shall not legislators and voters *know* that these things are calculable? Shall we go on with vague promise of oratorical delights without even a description of the way in which results are reached? Shall we again trust implicitly those "experts" who have so lately lost our confidence? Shall not the public know at least a little about the mathematical foundation on which sound decision rests?

Economic experts, themselves disillusioned by the failure of vaguely stated theories of quantitative economic concepts, are themselves turning to precise mathematical formulation of public economic questions. A large and growing section of the economists, headed by such men as Irving Fisher, Harold Hotelling, G. C. Evans, H. T. Davis and others, are accepting mathematical forms as the only reliable means of solving our economic problems. A new Society, called the Econometric Society, has been formed, and a new journal,

called *Econometrica*, is just being started, the first issue having appeared in January. These men, this Society, this journal, are formally sponsoring the mathematical formulation of public economic problems.

Another group—still more widely advertised—called the Technocratic group, is attacking these problems from a very different angle; but their methods also employ mathematical formulations and cannot possibly be understood by anyone who is ignorant of elementary algebra.

My mention of these two widely different efforts to solve pressing public problems is not in order to recommend either of them, much less to make propaganda for either of them. Rather my own purpose is to point out that both these groups, and other serious groups that are attempting to solve public problems, are tending more and more toward exact formulation of economic ideas, and toward the use of mathematical computations upon them.

To study such movements at all, even in order to criticize them intelligently, a considerable knowledge of mathematics is absolutely necessary. I should say again, however, that the mathematical knowledge necessary is *not* great; the main ideas can be understood by all who have taken high school algebra in a well-taught class. Any presentation of the plans of any such group or school of thought that does not contain at least some indications of the mathematical basis for the computations used, is admittedly only superficial, and cannot be the true ground for final acceptance of such a scheme, nor, indeed, for its rejection.

Even the more conservative leaders in business and in economics admit that some reorganization of business, of banking, of insurance, on industry, is necessary in order to restore any true economic prosperity. Such a restoration of a sound economic scheme is universally and devoutly desired by all thinking people. Whether it is to be a sound reorganization which will stand, or a superficial reorganization based on emotion and on vague promises that invite fresh disaster, depends upon the intelligent judgment of the whole public, upon the formation of an intelligent public opinion. A comprehension of the very meaning of the issues must precede any intelligent judgment; and for this some idea of the mathematical basis of the necessary computations is indispensable.

In the present emergency, men have learned to their sorrow that blind faith in "experts" must be supplemented by enough understanding of the underlying problems to enable the public as a whole to fore-

see and to prevent those abuses that have been all too common. No less than widespread understanding of the mathematics that lies behind all business and all economics is absolutely necessary for the proper reorganization of business and for the proper safeguarding of our commonwealth.

SAVING TIME IN CALCULATING ACTUAL INTEREST FROM INTEREST TABLES

By I. C. NICHOLS
Louisiana State University

The business world is ever sensitive to the "cost of production". But in figuring this cost the element of time necessarily plays an important part—often the major part. Consequently there is always the urge to do a thing in the shortest time possible commensurate with quality and efficiency.

Logically, then, one party to a transaction may accept an apparent loss respecting one feature of the deal through using a certain time-saving device with a feeling that the loss incurred is more than offset in the end by the value of the time thereby saved. An illustration of the above is found in the custom of figuring interest. In calculating the *actual* interest for fifty-seven days on a note for one thousand dollars at the rate of 8% per annum the amateur will find the interest to be \$12.50; the professional clerk will find it to be \$12.67. Why this difference of \$ .17? It is largely the result of the habit of saving time. If an actual year of 365 days be used, the interest is \$12.50; hence it is called the *actual* interest. If an *ordinary* year of 360 days be used, the interest is \$12.67; hence it is called the *ordinary* interest. The number 360 is a much more convenient number than 365. The only factors of 365 are 5 and 73, while 360 has quite a host of factors, convenient for cancellation and time-saving to professional clerks and calculators. Interest tables are based on a year of 360 days. Expressed as an equation, the *ordinary* interest I on a sum of money P at the rate i for d days is $I = Pid/360$; whereas the actual interest I'

under the same terms is $I' = Pid/365$. The ratio of I to I′ is then 73/72, that is,

$$I = (73/72)I' = I' + I'/72$$

or

$$I' = (72/73)I = I - I/73$$

Stated in words we have these simple rules:

(1) *Given the ordinary interest, the actual interest may be obtained by reducing it by* 1/73 *of itself*;

(2) *Given the actual interest, the ordinary interest may be obtained by increasing it by* 1/72 *of itself.*

The rules are simple to understand and to apply, and, certainly, they are time saving—the professional clerk may still use his interest tables. If the *ordinary* interest is desired, the answer is given by the tables; if the *actual* interest is desired, use the tables to find the *ordinary* interest and then diminish this sum by 1/73 of itself, the remainder being the *actual* interest sought.

The concluding suggestion is that High School and Grammar School teachers would do well to put their students wise to this simple rule so that they may apply it in business later in life. The rule is a practical and worthwhile one.

ON THE NATURE OF THE ROOTS OF A QUARTIC EQUATION

By RAYMOND GARVER

Criteria for the nature of the roots of the general quartic equation are well-known, and have been developed in many different ways. The matter, however, is certainly of considerable importance in the theory of equations, and I feel that a new presentation may be offered without apology provided it combines novelty and efficiency. I hope that the following treatment, based on a transformation of the given quartic, may seem to possess these two properties.

We shall consider the general quartic in the reduced form

(1) $$x^4+qx^2+rx+s=0,$$

and let D represent its discriminant, and F the quantity q^2-4s. It is a matter of simple algebra, based on the definition of the discriminant as the product of the squared differences of the roots of (1), to show that when $D=0$ (1) has a multiple root, and that when $D<0$ (1) has exactly two real roots. It is also easy to see that when $D>0$ the equation has either no real roots or four real roots; the difficulty lies in distinguishing between these possibilities. The criteria for accomplishing this may be written in two slightly different forms; where we have F, the slightly more complicated expression $8qs-2q^2-9r^2$ appears in some developments:

$D>0$, $q \not< 0$, (1) has no real roots,

(2) $D>0$, $F \not> 0$, (1) has no real roots,

$D>0$, $q<0$, $F>0$, (1) has four real roots.

Remember now that $D>0$ alone insures that the equation has no real roots or four distinct real roots. Hence if we can show the existence of even one real, or of one imaginary, root, then all four must be real, or imaginary, respectively. With this in mind, the first part of (2) is obtainable without difficulty. We need only the simple fact that $x_1^2+x_2^2+x_3^2+x_4^2=-2q$, where x_1, x_2, x_3, x_4, are the roots of (1). If the four roots are real and distinct q must obviously be negative.

The only difficulty in the whole problem lies in the proof of the last two parts of (2). Instead of employing any of the usual devices, such as Sturm's functions or the resolvent cubic, we shall use a transformation on the roots of (1) and Descartes' rule of signs. The transformation is $y=x^2+q/2$, and the transformed equation is set up just as the transformed equation for the transformation $y=x^2$ is often set up in elementary work. That is, we transpose the term rx to the right hand side of (1), replace x^2 by $y-q/2$, square both sides of the equation, and replace the x^2 which then appears on the right hand side by $y-q/2$. The transformed equation appears as

(3) $$f(y)=(y^2-F/4)^2-r^2(y-q/2)=0,$$

or, after simplifying, as

(4) $$y^4-Fy^2/2-r^2y+F^2/16+qr^2/2=0.$$

It should be remarked that there is not necessarily a one-to-one correspondence between real roots of (1) and real roots of (4), for, under the transformation $y=x^2+q/2$ both real and pure imaginary roots of the former go into real roots of the latter. However, (1) does not have more real roots than (4), and it will have exactly the same number when (4) does not have a double root, since a pair of conjugate pure imaginary roots of (1) transform into a double real root of (4).

Now if $F \ngtr 0$, Descartes' rule of signs allows (4) a maximum of 2 real roots, whether the constant term be positive, negative, or zero, provided only that r and F are not both zero. The reader can check this statement in a moment. Further, $r=F=0$ makes the left hand side of (1) a perfect square, and hence D zero; this possibility is then ruled out. Now since $D>0$ does not permit (1) to have exactly two real roots, and since $D\neq 0$, $F\ngtr 0$, allow a maximum of two real roots for (4) and consequently for (1), the second part of (2) is seen to follow.

One comment may be made in this connection. If $F\ngtr 0$ and the constant term of (4) is negative or zero, (4) has exactly two real roots, one positive and one negative or zero. Not being double roots, these must have come from real roots of (1). It follows logically that the conditions $D>0$, $F\ngtr 0$, $F^2/16+qr^2/2\ngtr 0$, must be inconsistent. As a matter of interest, this may be checked by noting that $F\ngtr 0$ requires s to be non-negative, that $F^2/16+qr^2/2\ngtr 0$ requires q to be non-positive, and that D may be written in the form $16s(F^2+8qr^2)-r^2(4qF+27r^2)$.

In case $q<0$, $F>0$, Descartes' rule of signs does not seem to be adequate. But now, using (3) for convenience instead of (4), we see that $f(\sqrt{F}/2)=-r^2(\sqrt{F}-q)/2\ngtr 0$. If the equality holds, r must be zero, or q and s must both be zero. If $r=0$ it is easily seen that (1) cannot have all of its roots imaginary when $q<0$ and $F>0$, for two of the roots of (1) are the roots of $x^2=(-q+\sqrt{F})/2$. If $s=q=0$, $r\neq 0$, (1) has exactly two real roots; this case cannot arise when $D>0$. When $f(\sqrt{F}/2)<0$, we note that $f(\infty)$ and $f(-\infty)$ are both positive, and conclude that (4), and hence (1), has at least two distinct real roots. The third part of (2) then follows, and the criteria are completely obtained.

ON EXHIBITS IN MATHEMATICS

By MARIE LOUISE RENAUD
Bay St. Louis, Miss.

Mathematics has long been in need of that popular interest and appeal which other subjects possess. In this day of wide-spread advertising we mathematics teachers should come in for our share. In schools where only one or two years of mathematics are required for graduation the need of creating more interest in Algebra and Geometry is greater. I have just finished presenting a Geometry exhibit for the High School students and members of the local P. T. A., and its success has made me realize that interest in mathematics is certainly not hard to arouse. The exhibit was not only of value to the students participating but of inestimable worth to the other high school students who are contemplating electing the subject. Before taking it so few students have any conception of the subject matter dealt with in Geometry.

Not having any bulletin board to use, I made one by flattening the four sides of a corrugated box. This gave a neat brown background. The exhibits consisted of original constructions, constructions found in the book, a study of words used in Geometry, excellent test papers, work-book exercises and drawings of polygons and angles. All students in the class were intensely interested in the exhibit and started work on it as soon as I had told them about it several weeks ago. Each member of the class was allowed to hand in as many originals as desired and the most artistic, neatest and accurate ones were selected for the display.

For several years I have had one or two exhibits each year and each time I realized their value cannot be estimated in the creating of popular interest in the subject. Despite the nonchalance of high school students they do get a "thrill" out of seeing their work on display. My Algebra classes have been asking me "When are *we* going to have one?" After all what are we Geometry teachers trying impart? If we succeed in making the student think for himself; if we succeed in developing his originality; if we give him the historical significance of the subject and make familiar to him the famous names in mathematical history, then indeed is Geometry justified in our school corriculum.

THE HYPERGEOMETRIC OF GAUSS

By ABE HACKMAN
Rensselaer Polytechnic Institute

Biographical Note

Karl Friedrich Guass, ranked with Laplace and Lagrange as one of the three greatest masters of modern mathematical analysis, was born in Brunswick on April 30, 1777. The son of a poor brick-layer, he distinguished himself at an early age as a brilliant student, and was enabled through the generosity of the then reigning Duke of Brunswick to continue his formal schooling at Caroline College and the University of Goettingen.

In 1807 he was called to Goettingen to become the director of the newly established astronomical observatory. He remained in Goettingen practically all his life and at his death in 1855 he still held the post of director of the observatory and Professor of Astronomy.

He is known for the pioneering work he did in the fields of electricity and magnetism, and for his researches in astronomy as well as for a large number of discoveries and investigations in pure mathematics.

His collected works in Latin comprise six large volumes. In them can be found discoveries in the realm of complex numbers, of number theory, of determinants, of the method of least squares and many others. The discussion of the Hypergeometric series is in about seventy pages and first appeared in 1812.

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1. The Hypergeometric which Gauss denoted by $F(\alpha,\beta,\ \gamma,x)$ is the series:

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,\gamma(\gamma+1)(\gamma+2)}x^3+\ldots(1)..$$

x is the variable of the series and α,β, and γ are known as the parameters of the series. These latter determine the function, if any, which the series represents.

The series is important because it is a general form of a power series, a series composed of powers of a variable each multiplied by a coefficient which is determined by some law.

2. The series as can be seen by inspection is an infinite series, that is one which has a number of terms greater than any finite number we may choose. But if α or β or both have the value 0, the series degenerates and becomes 1. If $\gamma=0$, the series cannot be used to represent any finite quantity, since all terms after the first become infinite. If α or β or both are equal to $-k$, where k is a positive integer, the series will no longer be an infinite series but will be a polynomial having $(k+1)$ terms. If $\gamma=-r$, where r is a positive integer, the $(r+1)$th term and all terms after it become infinite, and the series cannot be used to represent a finite quantity.

3. A highly important property of the series is the general ratio between one term and the preceding term. In a geometric series this ratio is the same whatever terms are considered. As will be seen, in the Hypergeometric, this ratio depends on the *number* or position of the terms as well as on the variable and the parameters.

Let us find the general or the mth term of the series. We see from (1) that the exponent of x in any term is one less than the number of the term, i. e.: in the first term 0, in the second, 1, etc. The products $\alpha(\alpha+1)\ldots$, $\beta(\beta+1)\ldots\ldots$, $\gamma(\gamma+1)\ldots\ldots$ are also seen to have as their last factor $\alpha+$(two less than the number of the term). . . . The factorial in the denominator has as its last factor the exponent of x. Or if we denote by w_m the mth term, we have:

$$w_m=\frac{\alpha(\alpha+1)\ldots(\alpha+m-2)\beta(\beta+1)\ldots(\beta+m-2)}{1\,.\,2\,.\,3\ldots(m-1)\,\gamma(\gamma+1)\ldots(\gamma+m-2)}\,x^{m-1}$$

$$w_{m+1}=\frac{\alpha(\alpha+1)\ldots(\alpha+m-1)\beta(\beta+1)\ldots(\beta+m-1)}{1\,.\,2\,.\,3\,.\,(m)\,.\,\gamma(\gamma+1)\ldots(\gamma+m-1)}\,x^{m} \qquad 2)$$

$$w_{m+2}=\frac{\alpha(\alpha+1)\ldots(\alpha+m)\;\beta(\beta+1)\ldots(\beta+m)}{1\,.\,2\,.\,3.\,.(m+1)\ldots\gamma(\gamma+m)}\,x^{m+1}$$

We choose as the terms to be used in finding the ratio, the (m+2)nd and the (m+1)st, since this ratio is somewhat simpler than that between the (m+1)st and the mth. Or

$$\frac{w_{m+2}}{w_{m+1}} = \frac{(\alpha+m)(\beta+m)}{(1+m)(\gamma+m)}\,x \tag{3}$$

a result which follows directly from (2).

This ratio is known as the *Test Ratio* of the Hypergeometric.

4. Gauss studied many properties of the Hypergeometric, notably its convergence. An infinite series has, as has been mentioned, an infinite number of terms. In general, the sum of its terms will not approach a finite quantity as more and more terms are taken into consideration. When the sum of a series does approach a finite limit as the number of its terms increases indefinitely, it is said to be convergent. Gauss discovered for what values of x, α, β, γ, the hypergeometric is convergent. This is extremely important, since the series can be used to represent a finite quantity only when convergent. In the discussion that follows, we shall assume that we are discussing each series, within such a range of values of its variable and parameters as to make the series convergent and permit the use of the equality sign (=) between the function and its associated series.

5. Once Gauss had discovered the conditions under which the Hypergeometric is convergent, it became possible to use these discoveries in determining the convergence of a particular power series, provided the particular series were placed in the Hypergeometric form, and its particular α, β, γ, and x were known.

6. Gauss himself gave 23 examples of how various well known series could be put into the form of the hypergeometric. We shall discuss several of the more simple ones.

7. Let us consider the series which may when convergent be used to represent log (1+t). This series is

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{5} - \frac{t^4}{4} + \dots \tag{4}$$

Comparing (4) with (1), we see first that they differ in that the first term of (4) is t, while the first term of (1) is 1. Therefore, the first step in Gaussifying (4) must be to divide it by t or

$$\log(1+t) = t[1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots]. \qquad (5)$$

Let us now consider the series in brackets in (5). If we find its test ratio and get it equal to the test ratio of the hypergeometric, we shall find what α, β, γ, and x are for the log of $(1+t)$. The general or kth term in the series in (5) is seen to be

$$w_k = \frac{(-1)^{k-1} t^{k-1}}{k} \text{ or } \frac{w_{m+2}}{w_{m+1}} = \frac{(-1)^{m+1} t^{m+1}}{(m+2)} \cdot \frac{(m+1)}{t^m(-1)^m}$$

$$= -\frac{(1+m)}{(2+m)} t \qquad (6)$$

or

$$\frac{(\alpha+m)(\beta+m)}{(1+m)(\gamma+m)} x = \frac{(1+m)}{(2+m)} (-t);$$

in order that these two ratios should be identical it is necessary that there be a $(1+m)$ factor in the denominator of the second. Or multiplying the second by $(1+m)$ and dividing by $(1+m)$ we have

$$\frac{(\alpha+m)(\beta+m)}{(1+m)(\gamma+m)} x = \frac{(1+m)(1+m)}{(1+m)(2+m)} (-t) \text{ or } \begin{array}{l} \alpha = 1 \\ \beta = 1 \\ \gamma = 2 \\ x = -t \end{array} \qquad (7)$$

or

$$\log(1+t) = tF(1, 1, 2, -t) \qquad (8)$$

8. Let us now consider the well known binomial expansion

$$(1+t)^n = 1 + nt + \frac{n(n-1)}{1 \cdot 2} t^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} t^3 + \ldots.$$

It corresponds to the Hypergeometric, in that both start with 1. Its general term using the method of generalization employed in § 3 is:

$$w_k = t^{k-1}\frac{n(n-1)\ldots[n-(k-2)]}{(k-1)!} = \frac{n(n-1)\ldots(n-k+2)}{(k-1)!}t^{k-1} \quad \text{or}$$

$$\frac{w_{m+2}}{w_{m+1}} = \frac{n(n-1)\ldots\ldots(n-m)}{(m+1)!}t^{m+1}.$$

$$\frac{(m)!}{n(n-1)\ldots(n-m+1)}\frac{1}{t^m} = \frac{(n-m)}{(1+m)}t \tag{9}$$

Comparing this with $\dfrac{(\alpha+m)(\beta+m)}{(1+m)(\gamma+m)}$ x, we see that signs must be manipulated to make the sign of m in (9) positive, and also that what corresponds to $(\gamma+m)$ must have cancelled what corresponds to *either* $(\alpha+m)$ or $(\beta+m)$. We therefore set $\gamma=\beta$, whatever β is, or

$$\frac{(\alpha+m)(\beta+m)}{(1+m)(\gamma+m)}x = \frac{(m-n)(\beta+m)}{(1+m)(\beta+m)}(-t) \text{ or } \begin{matrix}\alpha=-n\\ \beta=\gamma\\ x=-t\end{matrix} \quad \text{(or}$$

$$(1+t)^n = F(-n,\ \beta,\ \beta,\ -x). \tag{10}$$

From what has been said in section 2 about the conditions under which the hypergeometric becomes a polynomial having a finite number of terms, it follows that when n is a positive integer, or when $(1+t)$ is raised to a positive integral power, there will be $(n+1)$ terms in the polynomial, a result to be expected from elementary knowledge of the binomial expansion.

9. The last series that we shall Gaussify is e^t. This series is important not only in itself but also because from it are derived other important functions notably sin t, cos t, etc.

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots .$$

$$W_k = \frac{t^{k-1}}{(k-1)!} \quad \text{or,} \quad \frac{w_{m+2}}{w_{m+1}} = \frac{t^{m+1}}{(m+1)!}\,\frac{m!}{t^m} = \frac{1t}{(m+1)} \tag{11}$$

Comparing with the test ratio of the hypergeometric again, we have

$$\frac{(\alpha+m)(\beta+m)}{(1+m)(\gamma+m)}\,x = \frac{1}{(1+m)}\,t.$$

In the general ratio there are two factors in the denominator and two in the numerator. In the ratio for e^t there are none in the numerator and only one in the denominator. The $(1+m)$ factors correspond. Firstly then it must be necessary that $(\gamma+m)$ must have canceled either $(\alpha+m)$ or $(\beta+m)$. Let us choose $\alpha=\gamma$ and since they can equal anything since they cancel, let $\alpha=\gamma=1$.. Or

$$\frac{(\alpha+m)(\beta+m)}{(1+m)(\gamma+m)}\,x = \frac{1}{(1+m)}\,t, \text{ or } (\beta+m)x = t \;:\; x = \frac{t}{\beta+m}$$

Therefore $\quad \alpha=\gamma=1$

$$x = t/\beta$$

$$e^t = F(1,\ \beta,\ 1,\ t/\beta) \tag{12}$$

provided that in developing the series we do not treat the variable t/β as follows:

$$e^t = F(1,\ \beta,\ 1,\ t/\beta) = 1 + \frac{1\,.\,\beta}{1\,.\,\beta}\,\frac{t}{\beta} = \frac{1\,.\,2\,.\,\beta(\beta+1)}{1\,.\,2\,.\,1\,.\,2}\,.\,\frac{t^2}{\beta^2}$$

$$+ \frac{1\,.\,2\,.\,3\ \beta(\beta+1)(\beta+2)t^3}{1\,.\,2\,.\,3\,.\,1\,.\,2\,.\,3\ \ \beta^3} + \ldots .$$

in which case the equality would hold only if β were infinite, but rather consider the β in the denominator of the variable to act as the β in the numerator of each term as:

$$e^{t} = F(1,\ \beta,\ 1,\ t/\beta) = 1 + \frac{1\,.\,\beta}{1\,.\,1}\,\frac{t}{\beta} + \frac{1\,.\,2\,.\,\beta(\beta+1)}{1\,.\,2\,.\,1\,.\,2\,.}$$

$$\cdot\frac{t^2}{\beta(\beta+1)} + \frac{1\,.\,2\,.\,3\,.\,\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,1\,.\,2\,.\,3}\,\frac{t^3}{\beta(\beta+1)(\beta+2)} + \ldots.$$

10. The preceding illustrations of how a particular power series can be put into the general form of the Hypergeometric, by determining its α, β, γ, and x, makes the study of the conditions of absolute and uniform convergence of the Hypergeometric highly important and fertile. This study may be treated in a later paper.

ON TEACHING A UNIT IN PLANE GEOMETRY

By JAMES EDMUND CONGLETON
Canton, Miss.

To a few pupils demonstrative geometry seems almost evident. They experience no difficulty in applying the hypothesis to the figure; the conclusion is easily determined and stated; what is given and what is to be proved suggests the plan of proof; one step in the proof leads to the next; and the reason for each statement seems perfectly natural.

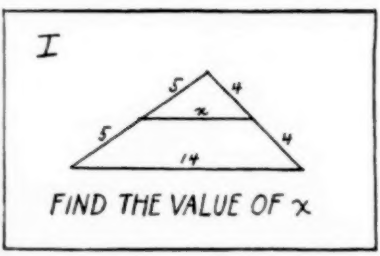
A great many pupils, however, find the study extremely difficult. They may not be able to determine the hypothesis unless it is stated in an "if clause"; the conclusion, even, may not be readily comprehended; the plan of proof for each theorem is learned by sheer force of memory; and the reason given for one angle beng greater than another may be that they are corresponding parts of congruent figures.

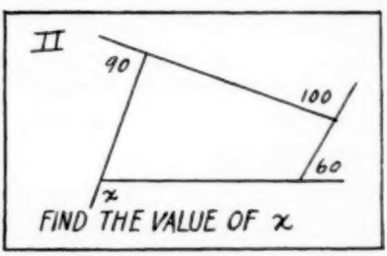
Unless the students in the second group receive a considerable amount of help during the class period, they will gain little by taking the course. They need careful explanations, drill over a limited amount of material, and encouragement, which can best be given by

setting a task that they can do. In trying to keep from failing an excessive number of geometry students and with the second group primarily in mind I have developed the following drills and devices.

I.

The generally accepted principle of dividing the material to be taught into units applies particularly well to the teaching of geometry. Beginning with the introduction and specimen proposition as the first unit, little difficulty will be had in dividing the book into units of suitable length. Perhaps there is one unit in the first book that many teachers have never discovered. When all the fundamental theorems in the first book have been completed, I usually have my classes spend a few days studying some theorems and exercises, the solution of which gives a numerical answer. This unit, which for want of a better name might be called "Numerical Relations in Book I", contains about 20 exercises, two of which are given as examples:





My students always enjoy this unit and I think it is worth while because it affords a new approach to a valuable body of material. These exercises embody a concrete problem and afford further drill in the fundamental ideas of geometry. In the two problems given as examples, as in all the other problems in this unit, the student must know at least one of the fundamental propositions.

II.

It may seem commonplace to say that all new terms should be carefully taught as they appear in the course. Every science carries

with it a new vocabulary, and to permit students to go through a course without learning this body of information is to condone their laziness and limit their possibilities. The study of new terms should emphasize understanding rather than sheer memory. Every possible approach to the new term should be utilized. The class should be shown that it is perfectly natural and reasonable that the term should be so named.

If the derivation of the word does not require too much knowledge of foreign languages, the approach should be made from that angle. After the student understands the meaning of the new term, he will need further drill to be able to express himself accurately and clearly. A very useful device for affording such drill can be had by compiling a list of questions which cover all the terms in the unit. These questions should be made so that it is possible to answer them with one word or a short phrase—the inclusive and exclusive part of the definition should be in the question itself. For example, instead of asking, "What are vertical angles?" ask, "What kind of angles are those which have the same vertex and the sides of one are prolongations of the sides of the other?" My students and I have compiled a list of 91 such questions for the first book and approximately the same number for the other books. These questions are not all concerned with the definition of terms. Sometimes a proposition or corollary can be so stated as to afford rapid drill and review: "To what is an exterior angle of a triangle equal?"

The teacher should not stop when the pupil is able to answer "vertical angles" to the question "What kind of angles are those which have the same vertex and the sides of one are the prolongations of the sides of the other?" The drill should be repeated until the student can answer the question, "What are vertical angles?" This list of of questions, then, will be helpful in the intermediate stage which consists of the interval between the time when a student understands a term or theorem and the time when he can give a clear expression of his idea.

Still another use is made of these questions. When a book is completed I give every student a copy of the complete list, instructing them not to write out the answers on the copy. Then one question on the next test or semester examination is to give the answers to certain questions on the list. This part of the test usually carries with it a value of about 20 per cent.

III.

One does not need to teach geometry very long before the discovery is made that it is not sufficient to have a proposition explained once. In trying to furnish adequate drill for my classes I have gone through a rather tortuous experience. At first, like nearly all inexperiened teachers, I expected too much of my pupils. I thought one explanation would be enough. The results of such a plan demanded a change. Then I decided to have each proposition in the assigned lesson explained several times during the class period by different pupils. Certain objections to such a procedure are obvious. It is monotonous. It encourages a pupil not to prepare his lesson since the chances of his being called on first are not very great. Then I found that the students could explain the proposition under consideration during the period in which it was up for discussion, but would make a complete failure of it on a test a few days later.

The method I am using at present seems to be effective When we come to a new proposition, I nearly always explain it thoroughly to the class, but, as I said before, explanation is not enough. Consequently during a class period I usually have students prove from three to five propositions and corollaries which have been proved before. If students are allowed to take their books to the board with them, they, knowing that they can copy the proof from the book if they are called on, will not study the propositions as they should. Therefore at the beginning of the period I let students go to the board and copy the theorems which are to be proved during that recitation. When these students sit down, I send another group, not allowing them to take their books, to the board to construct the figures and write out the proof.

This scheme has two distinct advantages. It sets a situation in which the pupil knows that he is going to be at the board with nothing to look to for help except the theorem, and by having certain propositions selected for special drill explained by several different students on successive days it affords plenty of drill on every proposition.

IV.

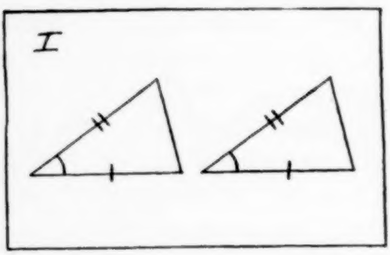
Not all students are required to learn to prove all propositions. It seems better to let them know that they will be able to make at least

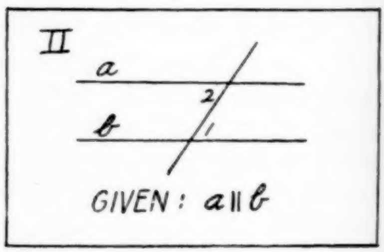
70 per cent without learning to prove some of the theorems which require superposition and indirect method or even some of those with difficult synthetic proofs. For every theorem, however, all students must be able to construct the figure, apply the theorem to it, state what is given, and what is to be proved.

Here is a good way to provide differentiated assignments. I would not be surprised to find that my students have their books marked so that they know which propositions are most likely to make up the body of the tests. Perhaps they mark them in some such manner as "study hard" or "omit". On nearly every test I give one or more of the propositions which have not received so much drill, and, in this way, can locate the students who deserve the high marks.

V.

Another device which I have found useful in teaching a unit in plane geometry consists of a group of figures indicating the theorems which the class is memorizing. Two figures which indicate theorems in the first book are given as examples:





In an exercise using this device students are given instructions to quote the theorems suggested by the figures. The first figure clearly indicates the theorem, "If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent". The second indicates, "If two parallel lines are cut by a transversal, the alternate angles are equal."

Figures can easily be drawn that will unmistakably indicate nearly all the theorems in the course. It would pay any teacher to

make a set of flash cards containing these drawings. This device proves useful on tests. Usually I give each member of my class a page of figures indicating all the propositions studied and instruct them not to write anything on the page since it will be used on the next test. The students bring the figures to class with them and I ask them to quote the propositions suggested by certain figures. I find this device has proved more satisfactory than any other to keep my students at work learning the theorems.

PROBLEM DEPARTMENT

Edited by
T. A. BICKERSTAFF
University, Miss.

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and solve problems here proposed.

Problems, and solutions will be credited to their authors.

Send all communications about problems to T. A. Bickerstaff, University, Mississippi.

Solutions

No. 26. Proposed by T. A. Bickerstaff:

Find the x's which represent digits in the following sum of a geometric progression:

$$4x+xxx+xxx+xxxx+xxxxx=xxxx1$$

Solution by the proposer and Mannis Charosh, 1901 84th Street, Brooklyn, New York.

Obviously the ratio Υ is rational and equal to say $\frac{a}{b}$. Since

$$\frac{4x}{b},\ \frac{4x}{b^2},\ \frac{4x}{b^3} \text{ and } \frac{4x}{b^4}$$

must be integral it follows that $b=1$ and Υ is integral.

Now,
$$\begin{aligned}
4x\ .\ \Upsilon^2 &< 1000 \\
\Upsilon^2 &< 25 \\
\Upsilon &< 5 \\
4x\ .\ \Upsilon^4 &> 9999 \\
49\Upsilon^4 &> 9999 \\
\Upsilon^4 &> 200 \\
\Upsilon &> 3 \quad \therefore \quad \Upsilon = 4
\end{aligned}$$

Now
$$4x\ \frac{1024-1}{4-1} = \text{xxxx1}$$

$$4x\ .\ 341 = \text{xxxx1}$$

If one factor ends in 1 and the product ends in 1, it follows that the other factor must end in 1. Hence the progression is:

$$41, +164, +656, +2624 + 10496 = 13981$$

No. 27. Proposed by H. T. R. Aude, Colgate University:

Show that the equation

$$2xy - px - py = 0$$

Where p is any prime number greater than 2 has three and only three positive integral solutions.

Solution by Mannis Charosh:

Let $2y-p=u$. Then $u=p^2/(2x-p)$.

Set $2x-p=\pm 1,\ \pm p,\ \pm p^2$ in turn and we find the only positive integral solutions are:

$$x=(p+1)/2\ ,\ p\ ,\ 0\ ,\ (p+p^2/2\ .$$
$$y=(p+p^2)/2\ ,\ p\ ,\ 0\ ,\ (p+1)/2\ .$$

If the 4th set is not considered distinct from the first we see that there are three sets of solution.

Problems for Solution

No. 30. Proposed by T. A. Bickerstaff:

A tells the truth $\frac{2}{3}$ of the time, B tells the truth $\frac{3}{4}$ of the time, C tells the truth on Sunday only. On a random day, A makes a statement which is confirmed by C but denied by B. What is the probability that it is true?

No. 31. Proposed by Rev. F. M. Kenney, Malone, N. Y.:

If n is prime and a is any integer, show that $\dfrac{a^{n-1}-1}{n}$ is an integer.

No. 32. Proposed by T. A. Bickerstaff:

One barrel contains a mixture of $\frac{2}{3}$ wine and $\frac{1}{3}$ water while a second contains a mixture of $\frac{1}{4}$ wine and $\frac{3}{4}$ water. How much must be drawn from each to fill a 2 gallon pail with a mixture of half wine and half water?

No. 33. Proposed by Rev. F. M. Kenney:

The trees in an orchard of rectangular form are arranged so that the rows are the same distance apart as the trees in the rows. Find the arrangements if half of the trees are on the outside, rows and ends.